RELATION BETWEEN DISLOCATIONS AND CONCENTRATED LOADINGS IN THE THEORY OF SHELLS

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Quantities are introduced which directly characterize the deformation of a normal element related to a given arbitrary line of the middle surface (in [7], [8] they were introduced for a line of curvature).

By means of these quantities the displacement vector is determined by using one contour integral (in contrast to [4]. This (by analogy with the procedure used in the plane problem) permits the construction of a theory of dislocations and the separation from the stress function of the nonsinglevalued parts related to the nonselfequilibrated loadings on the contours of the multiply connected regions. A dislocational approach to problems of the action of concentrated forces and moments on a shell is indicated.

1. The middle surface of the shell is referred to lines of principal curvature α_1, α_2. Thereby A_1, A_2, R_1, R_2 are Lamé parameters and the principal radii of curvature respectively. With regard to the boundary contour L, we assume that it consists of several closed smooth contours $L_j (j = 0, 1, \ldots, n)$, which may not coincide with the lines of principal curvature. Together with the basic system of unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_n\}$, we introduce a system [1], related to the given line $L'\{\mathbf{e}_\nu, \mathbf{e}_t, \mathbf{e}_n\}$ (Fig. 1). Obviously (Fig. 2),

$$(\gamma = \gamma(\alpha_1(s), \alpha_2(s)) = \mathbf{e}_1 \cdot \mathbf{e}_\nu) \qquad (1.1)$$

We also introduce

$$\frac{1}{R_\nu} = \frac{\cos^2\gamma}{R_1} + \frac{\sin^2\gamma}{R_2}, \qquad \frac{1}{R_t} = \frac{\sin^2\gamma}{R_1} + \frac{\cos^2\gamma}{R_2} \quad \frac{1}{R_{\nu t}} = \sin\gamma\cos\gamma\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (1.2)$$

We relate to the line L' the displacement vector

$$\mathbf{U} = \xi\mathbf{e}_\nu + \eta\mathbf{e}_t + w\mathbf{e}_n = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_n \qquad (1.3)$$

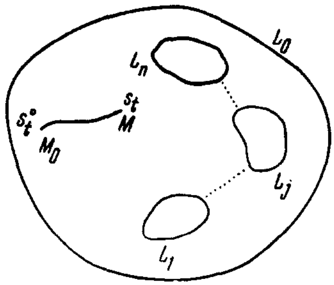

Fig. 1.

and the rotation vector

$$\Omega_t = -\omega_t e_\nu + \omega_\nu e_t - \omega_{t\nu} e_n = -\psi e_1 + \vartheta e_2 - \omega_{t\nu} e_n \tag{1.4}$$

It is not difficult to establish the following relations

$$\omega_\nu = \cos\gamma\,\vartheta + \sin\gamma\,\psi, \qquad \omega_t = -\sin\gamma\,\vartheta + \cos\gamma\,\psi$$
$$\omega_{t\nu} = \cos^2\gamma\,\omega_2 + \sin\gamma\cos\gamma\,(\varepsilon_2 - \varepsilon_1) - \sin^2\gamma\,\omega_1 \tag{1.5}$$

The deformation of a normal element, whose middle line is L', is characterized by the stretching of the middle line

$$\varepsilon_t = \cos^2\gamma\,\varepsilon_2 - \sin\gamma\cos\gamma\,\omega + \sin^2\gamma\,\varepsilon_1 \tag{1.6}$$

and the vector of change of curvature

$$\varkappa_t = -\varkappa_{tt} e_\nu + \varkappa_{t\nu} e_t - \varkappa_{tn} e_n \tag{1.7}$$

where

$$\varkappa_{tt} = \cos^2\gamma\,\varkappa_2 - 2\sin\gamma\cos\gamma\,\tau + \sin^2\gamma\,\varkappa_1 + \sin\gamma\cos\gamma\left(\frac{\omega}{R_t} - \frac{\varepsilon_2 - \varepsilon_1}{R_{\nu t}}\right)$$

$$\varkappa_{t\nu} = \sin\gamma\cos\gamma\,(\varkappa_2 - \varkappa_1) + (\cos^2\gamma - \sin^2\gamma)\,\tau - $$
$$- \frac{\sin\gamma\cos\gamma}{R_t}(\varepsilon_2 - \varepsilon_1) + \left(\frac{\sin^4\gamma}{R_1} - \frac{\cos^4\gamma}{R_2}\right)\omega$$

$$\varkappa_{tn} = -\frac{\cos\gamma}{A_1 A_2}\left(\frac{\partial A_2\varepsilon_2}{\partial\alpha_1} - \frac{\partial A_1\omega}{\partial\alpha_2} - \frac{\partial A_2}{\partial\alpha_1}\varepsilon_1\right) - \frac{\sin\gamma}{A_1 A_2}\left(\frac{\partial A_1\varepsilon_1}{\partial\alpha_2} - \frac{\partial A_2\omega}{\partial\alpha_1} - \frac{\partial A_1}{\partial\alpha_2}\varepsilon_2\right) +$$
$$+ \frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1}\left[\sin\gamma\cos\gamma\,(\varepsilon_1 - \varepsilon_2) - \cos^2\gamma\,\omega\right] -$$
$$- \frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2}\left[\sin\gamma\cos\gamma\,(\varepsilon_1 - \varepsilon_2) + \sin^2\gamma\,\omega\right]$$

Using the relations written down above and the known rules of differentiating unit vectors [2], we may establish two basic relations

$$\frac{\partial \mathbf{U}}{\partial s_t} = \omega_{t\nu}\mathbf{e}_\nu + \varepsilon_t\mathbf{e}_t - \omega_t\mathbf{e}_n = \varepsilon_t\mathbf{e}_t + \boldsymbol{\Omega}_t \times \mathbf{e}_t \tag{1.8}$$

$$\frac{\partial \boldsymbol{\Omega}_t}{\partial s_t} = \boldsymbol{\varkappa}_t \tag{1.9}$$

Here

$$\frac{\partial}{\partial s_t} = -\frac{\sin\gamma}{A_1}\frac{\partial}{\partial \alpha_1} + \frac{\cos\gamma}{A_2}\frac{\partial}{\partial \alpha_2}$$

is the derivative along the tangent with respect to L'. Thereby

$$\varkappa_{tt} = \frac{\partial \omega_t}{\partial s_t} + \left(\frac{\partial \gamma}{\partial s_t} + \frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\omega_\nu - \frac{\omega_{t\nu}}{R_{\nu t}},\quad \varkappa_{tn} = \frac{\partial \omega_{t\nu}}{\partial s_t} + \frac{\omega_t}{R_{\nu t}} + \frac{\omega_\nu}{R_t}$$

$$\varkappa_{t\nu} = \frac{\partial \omega_\nu}{\partial s_t} - \left(\frac{\partial \gamma}{\partial s_t} + \frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\omega_t - \frac{\omega_{t\nu}}{R_t} \tag{1.10}$$

We also have the following relationship:

$$\omega_\nu = -\frac{\partial \mathbf{U}}{\partial s_\nu}\cdot\mathbf{e}_n \qquad \left(\frac{\partial}{\partial s_\nu} = \frac{\cos\gamma}{A_1}\frac{\partial}{\partial \alpha_1} + \frac{\sin\gamma}{A_2}\frac{\partial}{\partial \alpha_2}\right) \tag{1.11}$$

indicating the derivative with respect to the normal of L'.

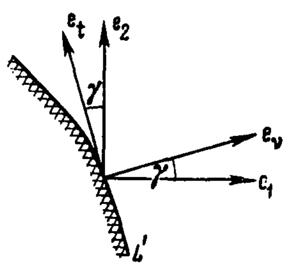

Fig. 2.

The following quantities are associated with the element considered: the traction vector

$$\mathbf{T}_\nu' = T_{\nu\nu}'\mathbf{e}_\nu + T_{\nu t}'\mathbf{e}_t + T_{\nu n}'\mathbf{e}_n \tag{1.12}$$

where

$$T_{\nu\nu}' = T_{\nu\nu} - \frac{M_{\nu t}}{R_{\nu t}} = T_1\cos^2\gamma + 2S\sin\gamma\cos\gamma + T_2\sin^2\gamma + \\ + \left(\frac{2H}{R_t} + \frac{M_1 - M_2}{R_{\nu t}}\right)\sin\gamma\cos\gamma$$

$$T_{\nu t}' = T_{\nu t} + \frac{M_{\nu t}}{R_t} = (\cos^2\gamma - \sin^2\gamma)S + \sin\gamma\cos\gamma(T_2 - T_1) + \\ + \frac{\sin\gamma\cos\gamma}{R_t}(M_2 - M_1) + \left(-\frac{\sin^4\gamma}{R_1} + \frac{\cos^4\gamma}{R_2}\right)2H$$

$$T_{\nu n}{}' = T_{\nu n} + \frac{\partial M_{\nu t}}{\partial s_t} = \frac{\cos\gamma}{A_1A_2}\left(\frac{\partial A_2M_1}{\partial\alpha_1} + \frac{\partial A_1 2H}{\partial\alpha_2} - \frac{\partial A_2}{\partial\alpha_1}M_2\right) +$$
$$+ \frac{\sin\gamma}{A_1A_2}\left(\frac{\partial A_1M_2}{\partial\alpha_2} + \frac{\partial A_2 2H}{\partial\alpha_1} - \frac{\partial A_1}{\partial\alpha_2}M_1\right) - \frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1}[\sin\gamma\cos\gamma\,(M_2 - M_1) +$$
$$+ \cos^2\gamma\, 2H] + \frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2}[\sin\gamma\cos\gamma\,(M_2 - M_1) - 2H\sin^2\gamma]$$

and the bending moment

$$M_\nu = M_1\cos^2\gamma + 2H\sin\gamma\cos\gamma + M_2\sin^2\gamma. \tag{1.13}$$

Let us express the displacement vector through the components of strain and rotation. From (1.8)

$$\mathbf{U} = \mathbf{U}_0 + \int\limits_{s_{t_0}}^{s_t} \{\varepsilon_t \mathbf{e}_t + \mathbf{\Omega}_t \times \mathbf{e}_t\}\, ds_t$$

Taking into account

$$\mathbf{e}_t = \frac{\partial\,(\mathbf{r} - \mathbf{r}_0)}{\partial s_t} \tag{1.14}$$

($\mathbf{r}$, $\mathbf{r}_0$ are respectively the radius vectors of the point L' and its original position), and taking into account (1.9), after integrating by parts we obtain

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{\Omega}_{t_0} \times (\mathbf{r} - \mathbf{r}_0) + \int\limits_{s_{t_0}}^{s_t} \{\varepsilon_t \mathbf{e}_t - \varkappa_t \times (\mathbf{r} - \mathbf{r}_0)\}\, ds_t + \left(\int\limits_{s_{t_0}}^{s_t} \varkappa_t ds_t\right) \times (\mathbf{r} - \mathbf{r}_0) \tag{1.15}$$

2. Following Love ([3], p. 232), we shall define as dislocations non-singlevalued displacements (in a multiply connected region) corresponding to unique components of strain. Therefore, assuming κ_t and ϵ_t to be singlevalued, we investigate the character of possible multivaluedness of displacements.

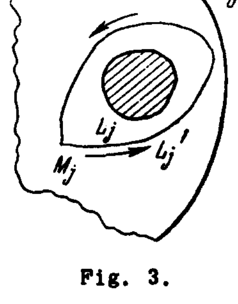

Fig. 3.

We consider contour L_j' originating and terminating (Fig. 3) at point M_j and embracing L_j. The initial value of the displacement vector is

$$U^- = U_0 + \Omega_{t_0} \times (r - r_j)$$

Following the contour L_j we may, in conformity with (1.15), return to M with a different value of a displacement vector:

$$U^+ = U_0 + \Omega_{t_0} \times (r - r_j) + \oint_{L'_j} \{\varepsilon_t e_t - \varkappa_t \times (r - r_j)\}\, ds_t + \Big(\oint_{L'_j} \varkappa_t ds_t\Big) \times (r - r_j)$$

Thus the increment of the displacement vector in the passage along L_j' is given by the expression

$$U^+ - U^- = U_j + \Omega_j \times (r - r_j) \tag{2.1}$$

The constants

$$U_j = \oint_{L_j'} \{\varepsilon_t e_t - \varkappa_t \times (r - r_j)\}\, ds_t, \qquad \Omega_j = \oint_{L_j'} \varkappa_t ds_t \tag{2.2}$$

will be called the parameters of dislocation.

We note that they do not depend on the shape of the contour which embraces L_j. This can easily be verified by considering the integral on the summary contour formed by two arbitrary contours L_j' and L_j'', which embrace L_j. Transforming the contour integral in the region enclosed by this contour, with the aid of (1.1), (1.6) and (1.7), we verify that the integrand will vanish by virtue of the continuity equations of the middle surface. This proves U_j and Ω_j to be independent of the shape of L_j'.

A fairly simple procedure may be advanced for the construction of the dislocational part of the displacement vector. To this end we consider the expression

$$\{U_j + \Omega_j \times (r - r_j)\}\, \Phi_j \tag{2.3}$$

With regard to $\Phi_j(\alpha_1, \alpha_2)$ we assume:

(1) that it gives in passing along L_j an increment of 1, and in passing along the remaining contours, zero;

(2) its derivatives in an arbitrary direction are singlevalued functions.

It is not difficult to verify that the expression (2.3) satisfies the requirements imposed on a dislocation. Examples of construction of dislocational functions will be considered below.

Problems regarding the analysis of shells from which a small portion is removed (or added) and then one edge is joined with the other by means of a rigid body displacement, lead to dislocational displacements. It is assumed thereby that the edges of the cut are congruent.

We shall also assume that the magnitude of the dislocation parameters are known. Then the dislocational displacement sought may be given the form

$$\mathbf{U} = \sum_j \{\mathbf{U}_j + \mathbf{\Omega}_j \times (\mathbf{r} - \mathbf{r}_j)\}\,\Phi_j + \mathbf{U}^{\circ\circ} \tag{2.4}$$

where $\mathbf{U}^{00}$ is the singlevalued part, permitting the boundary conditions and the surface loading which obtain in the problem considered to be met. From (1.8) follows

$$\varepsilon_t = \frac{\partial \mathbf{U}}{\partial s_t}\cdot \mathbf{e}_t, \qquad \omega_{t\nu} = \frac{\partial \mathbf{U}}{\partial s_t}\cdot \mathbf{e}_\nu, \qquad \omega_t = -\frac{\partial \mathbf{U}}{\partial s_t}\cdot \mathbf{e}_n$$

Using these relations, and also (1.10), (1.14) and the obvious equality $\partial(\mathbf{r} - \mathbf{r}_j)/\partial s_\nu = \mathbf{e}_\nu$, we obtain the expression for the components of deformation

$$\varkappa_{ttj} = -\mathbf{W}_j\cdot \mathbf{e}_n \frac{\partial^2 \Phi_j}{\partial s_t^2} + \left[-\mathbf{W}_j\cdot \frac{\mathbf{e}_t}{R_t} - 2\mathbf{\Omega}_j\cdot \mathbf{e}_\nu\right]\frac{\partial \Phi_j}{\partial s_t} - $$
$$-\mathbf{W}_j\cdot \mathbf{e}_n\left(\frac{\partial \gamma}{\partial s_t} + \frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\frac{\partial \Phi_j}{\partial s_\nu}$$

$$\varkappa_{t\nu j} = -\mathbf{W}_j\cdot \mathbf{e}_n \frac{\partial^2 \Phi_j}{\partial s_t \partial s_\nu} + \left[-\mathbf{W}_j\cdot \frac{\mathbf{e}_t}{R_t} + \mathbf{W}_j\cdot \frac{\mathbf{e}_\nu}{R_{\nu t}} - \mathbf{\Omega}_j\cdot \mathbf{e}_\nu\right]\frac{\partial \Phi_j}{\partial s_\nu} + $$
$$+\left[\left(\frac{\partial \gamma}{\partial s_t} + \frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\mathbf{W}_j\cdot \mathbf{e}_n - \mathbf{W}_j\cdot \frac{\mathbf{e}_\nu}{R_t} + \mathbf{\Omega}_j\cdot \mathbf{e}_t\right]\frac{\partial \Phi_j}{\partial s_t}$$

$$\varkappa_{tnj} = \mathbf{W}_j\cdot \mathbf{e}_\nu \frac{\partial^2 \Phi_j}{\partial s_t^2} + \left[\left(\frac{\partial \gamma}{\partial s_t} + \frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\mathbf{W}_j\cdot \mathbf{e}_t - \right.$$
$$\left. - 2\mathbf{\Omega}_j\cdot \mathbf{e}_n\right]\frac{\partial \Phi_j}{\partial s_t} - \mathbf{W}_j\cdot \frac{\mathbf{e}_n}{R_t}\frac{\partial \Phi_j}{\partial s_\nu}$$

$$\varepsilon_{tj} = \mathbf{W}_j\cdot \mathbf{e}_t \frac{\partial \Phi_j}{\partial s_t} \qquad (\mathbf{W}_j = \mathbf{U}_j + \Omega_j \times (\mathbf{r} - \mathbf{r}_j)) \tag{2.5}$$

We illustrate the statements made above with an example of a shell of revolution (Fig. 4), from which a portion has been removed; this portion was bounded by two close meridians and the edges subsequently brought into contact.

We have ([1], p. 241)

$$\alpha_1 = \theta, \quad \alpha_2 = \varphi, \quad A_1 = R_1(\theta), \quad A_2 = \rho = R_2(\theta)\sin\theta, \quad \frac{d\rho}{d\theta} = R_1\cos\theta$$

We put

$$\mathbf{U}_1 = 0, \qquad \mathbf{\Omega}_1 = \mathbf{\Omega}_{z1}\mathbf{e}_z, \qquad \Phi_1 = \frac{1}{2\pi}\varphi$$

Since

$$(\mathbf{r}-\mathbf{r}_1)=\rho\mathbf{e}_s+z\mathbf{e}_z, \qquad u^g=w^g=0, \quad v^g=\frac{1}{2\pi}\Omega_{z1}\rho\varphi \tag{2.6}$$

Finally, using (1.6) and (1.7), from (2.6) we obtain

$$\varepsilon_1=\omega=\varkappa_1=\tau=0, \qquad \varepsilon_2=\frac{1}{2\pi}\Omega_{z1}, \qquad \varkappa_2=\frac{1}{2\pi}\Omega_{z1}\frac{1}{R_2} \tag{2.7}$$

With the aid of these expressions, together with the relations combining the stress-resultants with the components of deformation, it is not difficult to obtain the expressions for the stress-resultants, and with these to determine the static boundary condition and the surface loading which correspond to displacements (2.6).

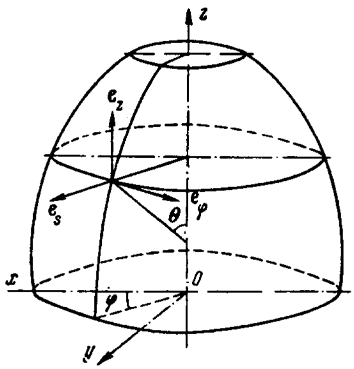

Fig. 4.

In the preceding discussion we made no assumptions regarding the magnitude of the contour L_j. Only its existence was of importance. From this it follows that the relations obtained will also be valid when L_j is shrinking to a point, if the close vicinity of the degenerate contour in which the quantities obtained may lose their significance and may become infinite is not considered.

3. Using the similarity of the equations expressing equilibrium and the continuity of the middle surface [2], we introduce the stress functions u^0, v^0, w^0 by means of the following differential relations:

$$\begin{aligned} T_1&=T_1^*+Ehc\varkappa_2^\circ(u^\circ, v^\circ, w^\circ), & M_1&=M_1^*-Ehc\varepsilon_2^\circ \\ T_2&=T_2^*+Ehc\varkappa_1^\circ, & M_2&=M_2^*-Ehc\varepsilon_1^\circ & \left(c=\frac{h}{\sqrt{12(1-\mu^2)}}\right) \\ S&=S^*-Ehc\tau^\circ, & H&=H^*+Ehc\frac{1}{2}\omega^\circ \end{aligned} \tag{3.1}$$

Here and later the superscript indicates that the quantity is constructed by means of the stress functions. Let T_1^*, T_2^*, S_1^*, M_1^*, M_2^* be a solution of the nonhomogeneous system of equilibrium equations.

It is not difficult to show that the expressions written down give the general solution of the system of equilibrium equations. Relations (3.1) generalize the Lur'e and Gol'denveizer functions [4,5] in a natural manner for the nonhomogeneous problem of the theory of shells. Using the functions thus introduced, with the aid of (1.12), (1.13), (1.6) and (1.7), the traction vector and the bending moment applied to a given contour may be given the following form:

$$\mathbf{T}_\nu' = \mathbf{T}_\nu'^* - Ehc\varkappa_t^\circ \qquad \begin{pmatrix} T_{\nu\nu}' = T_{\nu\nu}'^* + Ehc\varkappa_{tt}^\circ \\ T_{\nu t}' = T_{\nu t}'^* - Ehc\varkappa_{t\nu}^\circ \\ T_{\nu n}' = T_{\nu n}'^* + Ehc\varkappa_{tn}^\circ \end{pmatrix} \tag{3.2}$$

$$M_\nu = M_\nu^* - Ehc\,\varepsilon_t^\circ \tag{3.3}$$

The starred quantities correspond to the system of functions $(T_1^*, \ldots, H^*)$.

4. Let us consider the portion of the contour $s_{t0}s_t$ (Fig. 5). The principal traction vector, applied to the arc $s_{t0}s_t$ is

$$\mathbf{F} = \int_{s_{t0}}^{s_t} \mathbf{T}_\nu' ds_t = \int_{s_{t0}}^{s_t} \mathbf{T}_\nu'^* ds_t - Ehc \int_{s_{t0}}^{s_t} \varkappa_t^\circ ds_t$$

Recalling (1.9), we obtain

$$\mathbf{F} = \mathbf{F}^* - Ehc\,\{\boldsymbol{\Omega}_t^\circ - \boldsymbol{\Omega}_{t0}^\circ\} \tag{3.4}$$

The principal moment of all tractions and moments, applied to $s_{t0}s_t$ is

$$\mathbf{M} = \int_{s_{t0}}^{s_t} M_\nu \mathbf{e}_t ds_t + \int_{s_{t0}}^{s_t} (\mathbf{r}\times\mathbf{T}_\nu')\, ds_t =$$

$$= \int_{s_{t0}}^{s_t} M_\nu^* \cdot \mathbf{e}_t ds_t + \int_{s_{t0}}^{s_t} (\mathbf{r}\times\mathbf{T}_\nu'^*)\, ds_t - Ehc\left\{\int_{s_{t0}}^{s_t} \varepsilon_t^\circ \cdot \mathbf{e}_t ds_t + \int_{s_{t0}}^{s_t} (\mathbf{r}\times\varkappa_t^\circ)\, ds_t\right\}$$

Substituting κ_t by its expression from (1.9) in the last integral, and integrating by parts the integral thus obtained, and taking into account (1.8) and (1.14), we derive

$$\mathbf{M} = \mathbf{M}^* - Ehc\,\{(\mathbf{r}-\mathbf{r}_0)\times\boldsymbol{\Omega}_t^\circ + \mathbf{U}^\circ - \mathbf{U}_0^\circ\} \tag{3.5}$$

Solving (3.4) and (3.5) with respect to $\mathbf{U}^0$, we obtain

$$\mathbf{U}^\circ = \mathbf{U}_0^\circ + \Omega_{t0}^\circ\times(\mathbf{r}-\mathbf{r}_0) - \frac{1}{Ehc}\,\{(\mathbf{M}-\mathbf{M}^*) + (\mathbf{F}-\mathbf{F}^*)\times(\mathbf{r}-\mathbf{r}_0)\} \tag{3.6}$$

The last relationship is formally analogous to the displacement vector (1.15). Therefore, as was done for the dislocations, it may be shown that in the passage along L_j, the stress functions vector will receive an increment

$$\mathbf{U}^{\circ+} - \mathbf{U}^{\circ-} = \mathbf{U}_j^{\circ} + \mathbf{\Omega}_j^{\circ} \times (\mathbf{r} - \mathbf{r}_j) \qquad \left(\mathbf{U}_j^{\circ} = -\frac{\mathbf{M}_j - \mathbf{M}_j^*}{Ehc}\,,\ \mathbf{\Omega}_j^{\circ} = -\frac{\mathbf{F}_j - \mathbf{F}_j^*}{Ehc}\right) \tag{3.7}$$

Here $\mathbf{F}_j$ and $\mathbf{M}_j$ are the principal vector and the principal moment of all the tractions and moments applied at L_j.

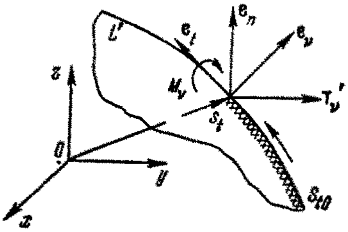

Fig. 5.

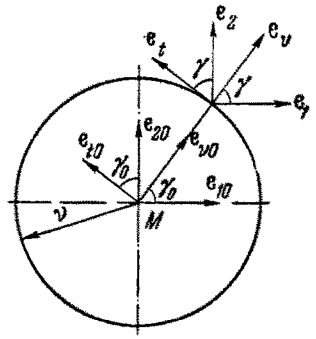

Fig. 6.

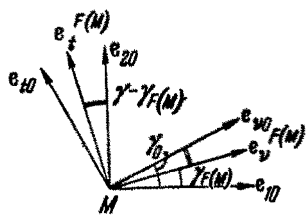

Fig. 7.

Further, an arbitrary stress functions vector may be represented in the form

$$\mathbf{U}^{\circ} = \Sigma_j \{\mathbf{U}_j^{\circ} + \mathbf{\Omega}_j^{\circ} \times (\mathbf{r} - \mathbf{r}_j)\}\, \Phi_j + \mathbf{U}^{\circ\circ\circ} \tag{3.8}$$

where Φ_j is the same dislocational function and $\mathbf{U}^{000}$ is the singlevalued part, which ensures the satisfaction of the equations of continuity with tractions and moments constructed by (3.8).

Let us assume that $\mathbf{F}_j$ and $\mathbf{M}_j$ are known. The quantities $\mathbf{F}_j^*$ and $\mathbf{M}_j^*$ should also be considered known, because they are constructed by means of the partial solution of the system of equations of equilibrium we have selected. Thus, $\mathbf{U}_j^0$ and $\mathbf{\Omega}_j^0$ are assumed to be known. The fundamental difficulty in finding the nonsingular part lies in the construction of the dislocational functions Φ_j. As soon as these are found, with the aid of (2.5), (3.2), (3.3) and (3.7), we obtain the following expressions for the tractions and moments:

$$T_{\nu\nu j}' = \mathbf{W}_j' \cdot \mathbf{e}_n \frac{\partial^2 \Phi_j}{\partial s_t^2} + \left[\mathbf{W}_j^{\circ} \cdot \frac{\mathbf{e}_t}{R_t} + 2(\mathbf{F}_j - \mathbf{F}_j^*) \cdot \mathbf{e}_\nu\right] \frac{\partial \Phi_j}{\partial s_t} +$$
$$+ \left(\frac{\partial \gamma}{\partial s_t} + \frac{\cos\gamma}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{\sin\gamma}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right) \mathbf{W}_j^{\circ} \cdot \mathbf{e}_n \frac{\partial \Phi_j}{\partial s_\nu}$$

$$T_{\nu t j}' = -\mathbf{W}_j^{\circ} \cdot \mathbf{e}_n \frac{\partial^2 \Phi_j}{\partial s_t \partial s_\nu} + \left[-\mathbf{W}_j^{\circ} \cdot \frac{\mathbf{e}_t}{R_t} + \mathbf{W}_j^{\circ} \frac{\mathbf{e}_\nu}{R_{\nu t}} - (\mathbf{F}_j - \mathbf{F}_j^*) \cdot \mathbf{e}_\nu\right] \frac{\partial \Phi_j}{\partial s_\nu} +$$

$$+\left[\left(\frac{\partial\gamma}{\partial s_t}+\frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)\mathbf{W}_j^\circ\cdot\mathbf{e}_n-\mathbf{W}_j^\circ\cdot\frac{\mathbf{e}_\nu}{R_t}+(\mathbf{F}_j-\mathbf{F}_j^*)\cdot\mathbf{e}_t\right]\frac{\partial\Phi_j}{\partial s_t}$$

$$T_{\nu nj}'=-\mathbf{W}_j^\circ\cdot\mathbf{e}_\nu\frac{\partial^2\Phi_j}{\partial s_t^2}+\left[-\left(\frac{\partial\gamma}{\partial s_t}+\frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)\mathbf{W}_j^\circ\cdot\mathbf{e}_t+\right.$$

$$\left.+2(\mathbf{F}_j-\mathbf{F}_j^*)\cdot\mathbf{e}_n\right]\frac{\partial\Phi_j}{\partial s_t}+\mathbf{W}_j^\circ\frac{\mathbf{e}_n}{R_t}\frac{\partial\Phi_j}{\partial s_\nu}$$

$$\mathbf{M}_{\nu j}=\mathbf{W}_j^\circ\cdot\mathbf{e}_t\frac{\partial\Phi_j}{\partial s_t}\qquad(\mathbf{W}_j^\circ=\{(\mathbf{M}_j-\mathbf{M}_j^*)+(\mathbf{F}_j-\mathbf{F}_j^*)\times(\mathbf{r}-\mathbf{r}_j)\})\qquad(3.9)$$

As before, the size of the internal contours is quite immaterial. If one of them (for example L_j) is shrunk to a point, conserving the magnitude of the principal vector and the principal moment, then (3.9) gives the values of the tractions and moments for a concentrated loading of given intensity. Let us investigate the character of the singularity of the solution (3.9) in the neighborhood of the application of concentrated loading (assuming that the point considered is not singular for the middle surface).

Let us introduce a local semigeodesic system of coordinates at the point ([6], p. 445). With an accuracy within small quantities of higher order for $\nu \ll 1$ (Fig. 6, 7) (the subscript j is omitted) we may obtain

$$\Phi\approx\frac{1}{2\pi}\gamma_0,\qquad\frac{\partial\Phi}{\partial s_t}\approx\frac{1}{2\pi}\frac{1}{\nu},\qquad\frac{\partial\Phi}{\partial s_\nu}\approx\frac{\partial^2\Phi}{\partial s_\nu\partial s_t}\approx\frac{\partial^2\Phi}{\partial s_t^2}\approx 0\qquad(\mathbf{r}-\mathbf{r}_0)\approx\nu\mathbf{e}_{\nu 0}$$

$$\mathbf{e}_\nu=\mathbf{e}_{\nu 0}+\nu\left\{\left[-\frac{\cos\gamma_0}{A_{10}A_{20}}\left(\frac{\partial A_1}{\partial\alpha_2}\right)_0+\frac{\sin\gamma_0}{A_{10}A_{20}}\left(\frac{\partial A_2}{\partial\alpha_1}\right)_0\right]\mathbf{e}_{t0}-\frac{\mathbf{e}_{n0}}{R_{\nu 0}}\right\}$$

$$\mathbf{e}_t=\mathbf{e}_{t0}+\nu\left\{\left[\frac{\cos\gamma_0}{A_{10}A_{20}}\left(\frac{\partial A_1}{\partial\alpha_2}\right)_0+\frac{\sin\gamma_0}{A_{10}A_{20}}\left(\frac{\partial A_2}{\partial\alpha_1}\right)_0\right]\mathbf{e}_{\nu 0}-\frac{\mathbf{e}_{n0}}{R_{\nu t0}}\right\}$$

$$\mathbf{e}_n=\mathbf{e}_{n0}+\nu\left\{\frac{\mathbf{e}_{\nu 0}}{R_{\nu 0}}-\frac{\mathbf{e}_{t0}}{R_{\nu t0}}\right\}$$

The index 0 indicates that the corresponding quantities are evaluated at the pole.

We resolve the principal vector and the principal moment into components tangential and normal to the surface:

$$\mathbf{F}=F_T\cdot\mathbf{e}_\nu^F+F_n\mathbf{e}_{n0},\qquad\mathbf{M}=M_T\cdot\mathbf{e}_\nu^M+M_n\mathbf{e}_{n0}$$

Obviously,

$$\mathbf{e}_{\nu 0}=\cos(\gamma_0-\gamma_F)\,\mathbf{e}_\nu^F+\sin(\gamma_0-\gamma_F)\,\mathbf{e}_t^F=\cos(\gamma_0-\gamma_M)\,\mathbf{e}_\nu^M+\sin(\gamma_0-\gamma_M)\,\mathbf{e}_t^M$$

$$\mathbf{e}_{t0}=-\sin(\gamma_0-\gamma_F)\,\mathbf{e}_\nu^F+\cos(\gamma_0-\gamma_F)\,\mathbf{e}_t^F=-\sin(\gamma_0-\gamma_M)\,\mathbf{e}_\nu^M+\cos(\gamma_0-\gamma_M)\,\mathbf{e}_t^M$$

With the aid of these relations, retaining in (3.9) only the terms with singularities, we obtain

$$\mathbf{T}_\nu' \sim \frac{1}{2\pi}\{\boldsymbol{\alpha}^T F_T + \boldsymbol{\alpha}^n F_n + \boldsymbol{\beta}^T M_T + \boldsymbol{\beta}^n M_n\}$$

where

$$\boldsymbol{\alpha}^T = \frac{2\cos^2(\gamma_0-\gamma_F)}{\nu}\mathbf{e}_\nu{}^F + \frac{\sin 2(\gamma_0-\gamma_F)}{\nu}\mathbf{e}_t{}^F, \qquad \boldsymbol{\alpha}^n = \frac{1}{\nu}\mathbf{e}_{n0} \tag{3.10}$$

$$\boldsymbol{\beta}^T = \frac{\cos 2\gamma_0}{\nu}\left(\frac{1}{R_{10}}-\frac{1}{R_{20}}\right)\mathbf{e}_t{}^M + \left\{\frac{\sin(\gamma_0-\gamma_M)}{\nu^3} - \right.$$

$$\left. - \frac{1}{\nu}\left[\frac{\cos(2\gamma_0-\gamma_M)}{A_1A_2}\left(\frac{\partial A_2}{\partial \alpha_1}\right)_0 + \frac{\sin(2\gamma_0-\gamma_M)}{A_{10}A_{20}}\left(\frac{\partial A_1}{\partial \alpha_2}\right)_0\right]\right\}\mathbf{e}_{n0}$$

$$\boldsymbol{\beta}^n = \left\{-\frac{\sin(\gamma_0-\gamma_M)}{\nu^2} + \frac{1}{\nu}\left[\frac{\cos(2\gamma_0-\gamma_M)}{A_{10}A_{20}}\left(\frac{\partial A_1}{\partial \alpha_2}\right)_0 - \frac{\sin(2\gamma_0-\gamma_M)}{A_{10}A_{20}}\left(\frac{\partial A_2}{\partial \alpha_1}\right)_0\right]\right\}\mathbf{e}_\nu{}^M +$$

$$+ \left\{\frac{\cos(\gamma_0-\gamma_M)}{\nu^2} + \frac{1}{\nu}\left[\frac{\sin(2\gamma_0-\gamma_M)}{A_{10}A_{20}}\left(\frac{\partial A_1}{\partial \alpha_2}\right)_0 + \frac{\cos(2\gamma_0-\gamma_M)}{A_{10}A_{20}}\left(\frac{\partial A_2}{\partial \alpha_1}\right)_0\right]\right\}\mathbf{e}_t{}^M$$

$$M_\nu = \frac{1}{2\pi}\left\{-\frac{\sin(\gamma_0-\gamma_M)}{\nu}M_T\right\} \tag{3.11}$$

If we retain only the principal singularities in expressions (3.10), then

$$2\pi\mathbf{T}_\nu' \sim \left[\frac{2\cos^2(\gamma_0-\gamma_F)}{\nu}\mathbf{e}_\nu{}^F + \frac{\sin 2(\gamma_0-\gamma_F)}{\nu}\mathbf{e}_t{}^F\right]F_T + \frac{1}{\nu}\mathbf{e}_{n0}F_n +$$

$$+ \frac{\sin(\gamma_0-\gamma_M)}{\nu^2}\mathbf{e}_{n0}M_T + \left[-\frac{\sin(\gamma_0-\gamma_M)}{\nu^2}\mathbf{e}_\nu{}^\mu + \frac{\cos(\gamma_0-\gamma_M)}{\nu^2}\mathbf{e}_t{}^M\right]M_n \tag{3.12}$$

From (3.11) and (3.12) it may be seen that the character of the principal singularity does not depend on the form of the middle surface or the choice of coordinates.

4. All the above is of a static-geometrical character and does not depend on the form of the relations between the tractions and moments and the components of strain of the middle surface. We have considered only such singularities of the solution as are static. In particular, the well-known logarithmic singularities are lacking.

The latter will occur if, assuming a generalized Hooke's law, we determine the periodic part of the stress functions vector $\mathbf{U}^{000}$.

A detailed study of this question requires the use of additional mathematical apparatus, and will be presented in a separate paper.

We shall here confine ourselves to the consideration of a simple illustrative example to clarify what has been said above. Consider a circular plate subjected to a central normal load P (Fig. 8). We have

$$\frac{1}{R_1}=\frac{1}{R_2}=0, \quad \alpha_1=\rho, \quad \alpha_2=\varphi, \quad A_1=r_0, \quad A_2=r_0\rho \quad \left(\rho=\frac{r}{r_0}\right)$$

Further

$$\mathbf{F}_1=P\cdot\mathbf{e}_n, \quad \mathbf{M}_1=0, \quad (\mathbf{r}-\mathbf{r}_1)=r_0\rho\mathbf{e}_\rho$$

$$\mathbf{F}_1\times(\mathbf{r}-\mathbf{r}_0)=Pr_0\rho\mathbf{e}_\varphi, \qquad \mathbf{W}_1^\circ=Pr_0\rho\mathbf{e}_\varphi$$

$$\Phi=\frac{1}{2\pi}\varphi, \quad \frac{\partial\Phi}{\partial\varphi}=\frac{1}{2\pi}, \quad \frac{\partial\Phi}{\partial\rho}=\frac{\partial^2\Phi}{\partial\rho\partial\varphi}=\frac{\partial^2\Phi}{\partial\varphi^2}=0$$

we have assumed $\mathbf{F}_1^*=\mathbf{M}_1^*=0$. Further, letting $\gamma=0$ and $\gamma=1/2\,\pi$, from expressions (3.9) we obtain

$$M_1=\frac{1}{2\pi}P, \quad T_1=T_2=S=M_2=H=0 \quad \left(N_1'=\frac{1}{2\pi r_0\rho}P, \quad N_2'=0\right)$$

Let us now determine the periodic part of the stress vector $\mathbf{U}^{000}$. From the character of the state of stress it follows

$$u^{\circ\circ\circ}=u^{\circ\circ\circ}(\rho), \qquad v^{\circ\circ\circ}=w^{\circ\circ\circ}=0$$

The total bending moments take on the form

$$M_1=\frac{P}{2\pi}-\frac{Ehc}{r_0}\frac{1}{\rho}u^{\circ\circ\circ}, \qquad M_2=-\frac{Ehc}{r_0}\frac{du^{\circ\circ\circ}}{d\rho} \tag{4.1}$$

The equation of continuity will be written down in the following form:

$$\rho\frac{d(M_2-\mu M_1)}{d\rho}+(1+\mu)(M_2-\mu M_1)=0$$

Substituting the expressions for the moments (4.1), we obtain a differential equation for the determination of u^{000}:

$$\rho^2\frac{d^2u^{\circ\circ\circ}}{d\rho^2}+\rho\frac{du^{\circ\circ\circ}}{d\rho}-u^{\circ\circ\circ}=\frac{(1+\mu)P}{2\pi}\rho\Big/\left(-\frac{Ehc}{r_0}\right)$$

Its particular solution has the following form:

$$u^{\circ\circ\circ}=\frac{P}{4\pi}(1+\mu)\rho\ln\rho\Big/\left(-\frac{Ehc}{r_0}\right)$$

Using it, we obtain from (4.1) the well-known expressions (see [9]) for the bending moments:

$$M_1=\frac{P}{2\pi}\left\{\frac{1+\mu}{2}\ln\rho+1\right\}, \qquad M_2=\frac{P}{2\pi}\left\{\frac{1+\mu}{2}\ln\rho+\frac{1+\mu}{2}\right\}$$

Thus the treatment of a simple example by means of the dislocational procedure has permitted all the singularities of the solution to be isolated.

In the general case the problem is reduced to finding particular solution of the system of equations in terms of complex displacements [8],

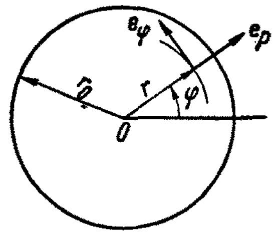

Fig. 8.

in which the quantities $(T_1^*,\ T_2^*,\ S^*,\ M_1^*,\ M_2^*,\ H^*)$ are constructed with the aid of the method discussed here.

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